Practice Problem Sheet II – Fall 2024

1. Given
$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m},$$
you can write the surface area of the energy shell at E in a microcanonical (MC) ensemble as
$$\Omega(E) = \frac{1}{h^{3N}N!} \int\int d^3x_1 \ldots d^3x_N \, d^3p_1 \ldots d^3p_N \, \delta\left(E - \sum_{i=1}^{N} \frac{p_i^2}{2m}\right).$$
Evaluate the integral above in MC ensembles. (Hint: This is an ideal gas problem in MC)

2. A Hamiltonian (for a two-level system) can be written as
$$H = -h \sum_{i=1}^{N} \sigma_i, \quad \sigma_i = \pm 1.$$
a) Show that in microcanonical ensembles, by writing,
$$\Omega(E) = \sum_{i=1}^{N} \delta(E + h \sum_{\sigma_i = \pm 1} \sigma_i) = 2^N \int \frac{dk}{2\pi} e^{f(k)}$$
one obtains an expression of $f(k)$,
$$f(k) = \imath kE + N \ln \cos kh. \qquad (\imath^2 = -1)$$
b) Compute the integral by using the saddle-point approximation and show that
$$S = k_B \ln \Omega(E) = -\frac{k_B N}{2}\left[(1+x)\ln\frac{1+x}{2} + (1-x)\ln\frac{1-x}{2}\right]$$
where $x = E/(Nh)$.
c) Calculate the temperature T from the definition of entropy and plot T as a function of x. Discuss your results for $x \geq 0$.

3. Show that in canonical ensembles the specific heat C_v is given by,
$$C_v = k_B \beta^2 \left[\langle E^2 \rangle - \langle E \rangle^2\right].$$

4. The partition function of a system in canonical ensembles is given by
$$Z = \frac{x}{1-x}, \text{ where } x = e^{-\frac{\beta\hbar\omega}{2}}.$$
Calculate U, C_v, F and S for this system.

5. A system consists of two energy levels $-\Delta/2$ and $\Delta/2$. Write down the the partition function of the system in canonical ensembles and calculate the specific hear C_v